SE3310 Pumping Lemma Study Guide

Pumping Lemma Recap

Recall the pumping lemma for regular languages states that for all strings in L longer than the pumping length (p), there exists *some* way to cut s into 3 parts xyz such that:

1. For all $i \geq 0, xy^iz \in L$
   - Explanation: the middle part y can be pumped (i.e., repeated arbitrarily many times, or even removed) and the resulting string is also in L
2. $|y| > 0$
   - Explanation: y, the loop portion, is non-empty. If y was empty, then pumping y wouldn't do anything!
3. $|xy| \leq p$
   - Explanation: p is defined to be the length after which a loop must occur in the corresponding DFA. x represents the portion of the string generated on the way leading up to the entrance of the loop, and y represents the portion of the string generated by the loop itself. Thus xy represents the string generated by a path through the DFA that contains a loop, and thus $|xy| \leq p$. This property can be useful in some languages as a means to make the proof simpler. It can allow the prover to reducing the number of possible cases for what y can contain.

Proof Steps

When performing the pumping lemma for regular languages, all proofs proceed with similar steps. Suppose we are asked to prove a language L is not regular.

1. Toward a contradiction assume L is regular
2. Let p be the pumping length
3. Pick a string s such that $s \in L$ and $|s| \geq p$
   - Note: The choice of s is left up to the prover, so pick a string that you think will

lend itself well to forming the contradiction

4. We must now examine all possible ways to divide s into 3 parts. Let $s = xyz$. We have to consider all possible cases for what might be contained in y.
   - Tip: From the pumping lemma we already know 2 things about y right out of the gate: $|y| \geq 0$ and $|xy| \leq p$ and thus $|y| \leq p$.
5. For each case identified in the previous step we must find our contradiction. That is, we must find a value $i \geq 0$ such that $xy^iz \notin L$.
   - Note: we only need to find one value for i in order to obtain the contradiction (since item 1 of the pumping lemma says it should hold for all $i \geq 0$).
6. Assuming we could successfully find a contradiction (step 5) for all possible cases (step 4) we conclude there is *no* way to divide s into 3 parts xyz such that y can be pumped. Therefore we conclude L is not regular.

An Example

Let $L = a^n b^n \ : n > 0$. Prove L is not regular.

Proof: We can directly follow the 6 proof steps outlined above.

1. Assume L is regular
2. Let p be the pumping length
3. Pick candidate string. Suppose we choose $s = a^p b^p$. Clearly $s \in L$ and $|s| = 2p$ and thus $|s| \geq p$.
4. Show pumping leads to a contradiction for all possible cases of what y could contain. First we consider the case where y contains only a's. There are no other cases since s is a string consisting of p a's followed by p b's, and as per the pumping lemma $|xy| \leq p$, meaning the xy portion of the string cannot include any b's.
5. Let $s = xyz$. Suppose we pump y up. Let $s_2 = xy^2z$, i.e., $xyyz$. Since y is non-empty, it means we added some non-zero number of a's to the string. Therefore if $s = xyz$ contained equal a's followed by b's, then $s_2 = xyyz$ must contain *more* a's than b's. But L is the language of strings that contain a's followed by an *equal* number of b's. Therefore $s_2 \notin L$. Therefore if L is regular, there must be some *other* way to divide s into 3 parts such that the middle part can be pumped. So we must examine any other cases for what y could contain. But there are no other cases.

6. According to the pumping lemma, if L is regular there must be *some* way to divide s

into 3 parts xyz such that $xyyz$ is also in the language. But we showed that wasn't possible. This is a contradiction. Therefore L is not regular.